Small Open Economy

Macroeconomic Theory, Lecture 9

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February 2018

Course material

Readings for lecture 9:

- Schmitt-Grohé and Uribe ("Open Economy Macroeconomics" online manuscript), Ch.1-5 (esp. 4)
- Schmitt-Grohé and Uribe (JIE, 2003), "Closing Small Open Economy Models"
- McCandless (2008), Ch. 13
- Bernanke and Gertler (AER, 1999), "Agency Costs, Net Worth, and Business Fluctuations"
- Bernanke, Gertler and Gilchrist (Handbook of Macroeconomics, Ch.21, 1989), "The financial accelerator in quantitative business cycle framework"

Small open economy RBC model - preliminaries

- Intertemporal models of current account provide foundations for open economy policy analysis
- Intertemporal approach views the current account (CA) balance as an outcome of dynamic *saving and investment* decisions of representative households. In particular, it recognizes that these decisions are based on expectations of future productivity growth, government spending demands, real interest rates, exchange rates...
- Intertemporal approach to CA combines elasticities and absorption approach view to CA: it accounts for importance of relative price changes *and* saving-investment balance as an explanation of CA movements.

Small open economy RBC model - preliminaries, cont.

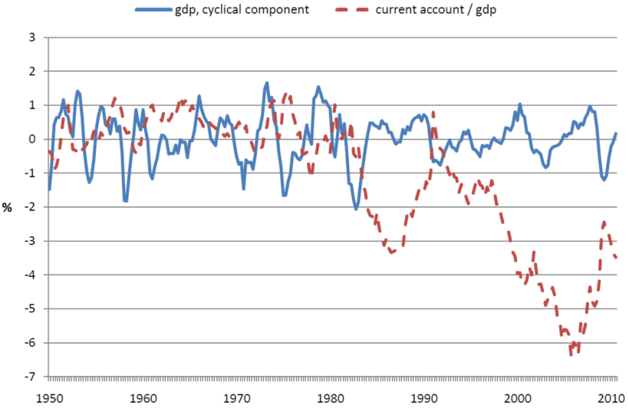
- Let B_t denote the economy's stock of net foreign assets at the end of period t, Y_t domestic output, C_t consumption, and I_t is investment
  - If $B_t < 0$, then the country has a net foreign debt $D_t = -B_t > 0$.
- CA is

$$CA_t \equiv \underbrace{B_{t+1} - B_t}_{\substack{\text{change in} \\ \text{net foreign} \\ \text{assets}}} = \underbrace{r_t B_t}_{\substack{\text{capital income} \\ \text{from net} \\ \text{foreign assets}}} + \underbrace{Y_t - C_t - I_t}_{\text{trade balance}} = S_t - I_t \tag{1}$$

Some empirical observations

- The current account can be countercyclical or procyclical
- In the US, the current account has been countercyclical
  - largest current account deficits during boom periods
- Procyclical current account: e.g. many oil exporting countries
- Finland: countercyclical current account in the 1970s and 1980s but procyclical current account since early 1990s

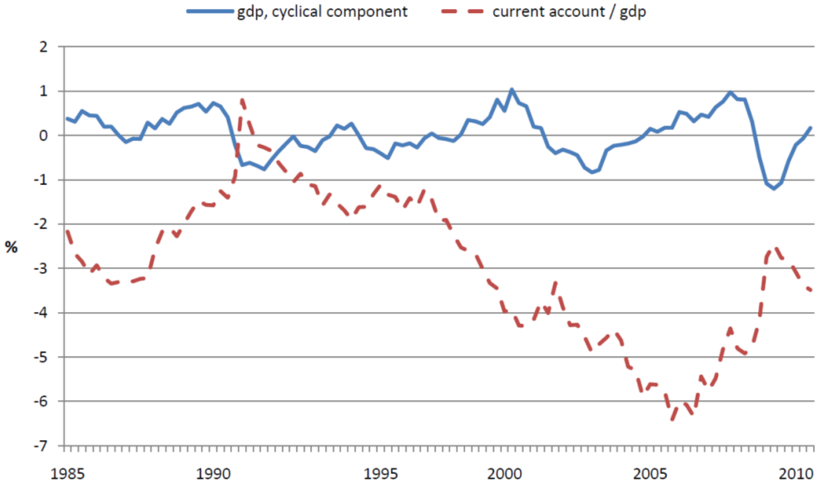
Current account and the business cycle in the US



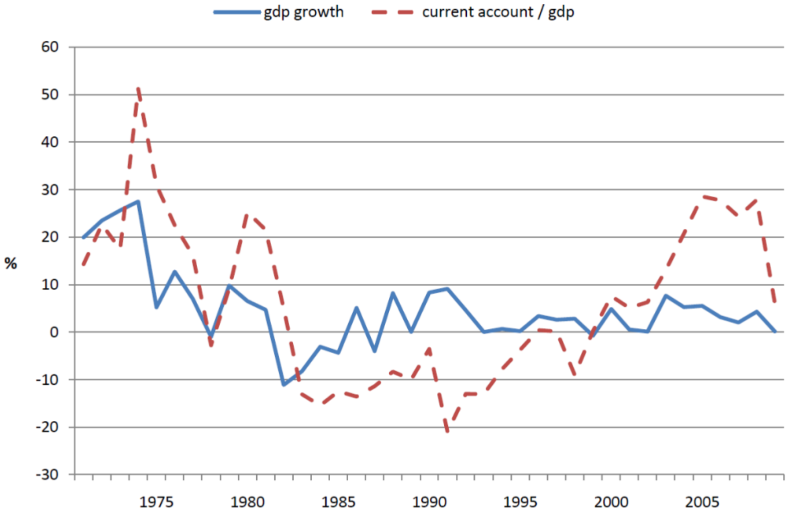
- $corr(GDP\_cycle, CA/GDP)$

1950$Q1$: 2010$Q2$	1950$Q1$: 1984$Q4$	1985$Q1$: 2010$Q2$
-0.11	-0.08	-0.27

Current account and the business cycle in the US, cont.

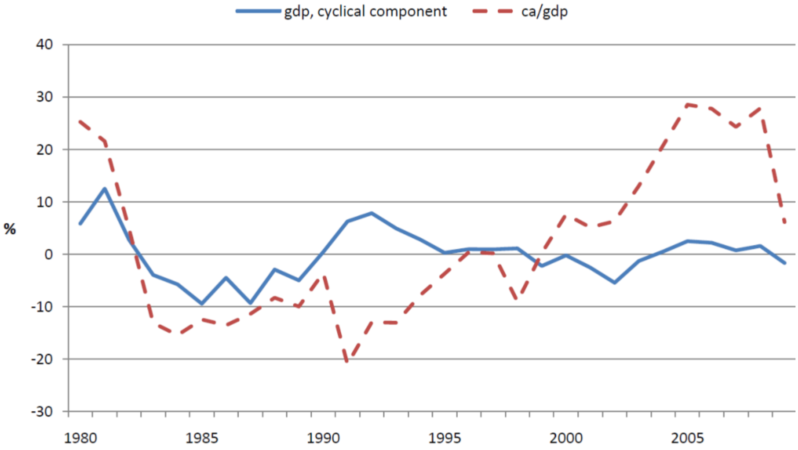


Current account and the business cycle in Saudi Arabia



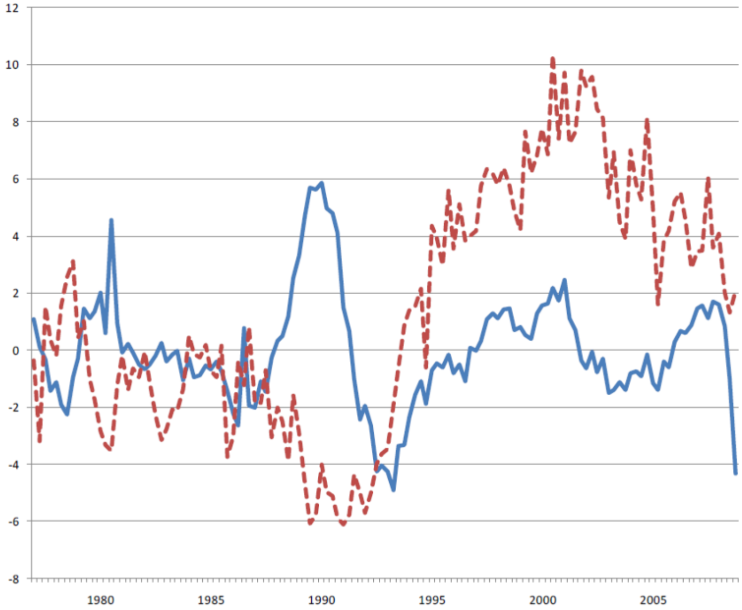
- Saudi Arabia $corr(\Delta \ln GDP,\ CA/GDP) = \ 0.54 \quad (1972-2009)$

Current account and the business cycle in Saudi Arabia



- $corr(GDP\_cycle,\ CA/GDP) = 0.34 \quad (1980 - 2009)$

Current account and the business cycle in Finland



Stylized facts: Fernandez and Gulan (2015)

Table : Emerging and developed markets business cycle moments.

Second moment[a]	Emerging markets		Developed markets	
$\sigma(Y)^b$	3.32^c	(0.27)	1.68	(0.19)
$\sigma(C)/\sigma(Y)$	1.26	(0.07)	0.65	(0.02)
$\sigma(I)/\sigma(Y)$	3.76	(0.40)	2.44	(0.11)
$\sigma(TB)$	3.21	(0.35)	1.29	(0.09)
$\rho(TB, Y)$	−0.40	(0.06)	0.33	(0.04)
$\rho(C, Y)$	0.77	(0.05)	0.58	(0.04)
$\rho(I, Y)$	0.69	(0.04)	0.63	(0.05)
$\sigma(R)$	0.92	(0.06)	0.35	(0.03)
$\rho(R, Y)$	**−0.36**	(0.06)	**0.17**	(0.07)

[a] All series were logged (except for TB), and then HP filtered. GMM applied to unbalanced panels. Data: 12 countries each group, 4Q 1993 – 3Q 2010.

[b] σ denotes standard deviation, ρ denotes correlation coefficient.

[c] Standard deviations expressed in %. Standard errors reported in brackets.

Small open economy RBC model and the current account

- A simple endowment economy (without capital accumulation) predicts procyclical current account balance:
  - current account plays a role of shock absorber. Households borrow to finance bad income shocks and save in response to positive income shocks. Increasing (decreasing) borrowing leads into deterioration (improvement) of current account balance.
- Sufficiently persistent productivity shocks can explain countercyclical behavior of the current account in an economy *with* capital accumulation (even without labor leisure choice.)
  - positive productivity shock
    - output goes up
    - investment grows more than domestic saving $\Rightarrow$ CA deficit
- But we need capital adjustment costs to make sure that investment does not respond too strongly (especially to changes in foreign interest rate).

Example: "The cycle is the trend"

- Aguiar and Gopinath (JPE, 2007) consider shocks with permanent effects, i.e. "trend shocks":

$$Y_t = A_t K_t^{\alpha} \left(\Gamma_t L_t\right)^{1-\alpha}$$

where $\frac{\Gamma_t}{\Gamma_{t-1}} = g_t$ and

$$\ln g_t = (1-\rho_g) \ln \bar{g} + \rho_g \ln g_{t-1} + \epsilon_{g,t}$$

Note that these shocks have permanent effect on the economy:

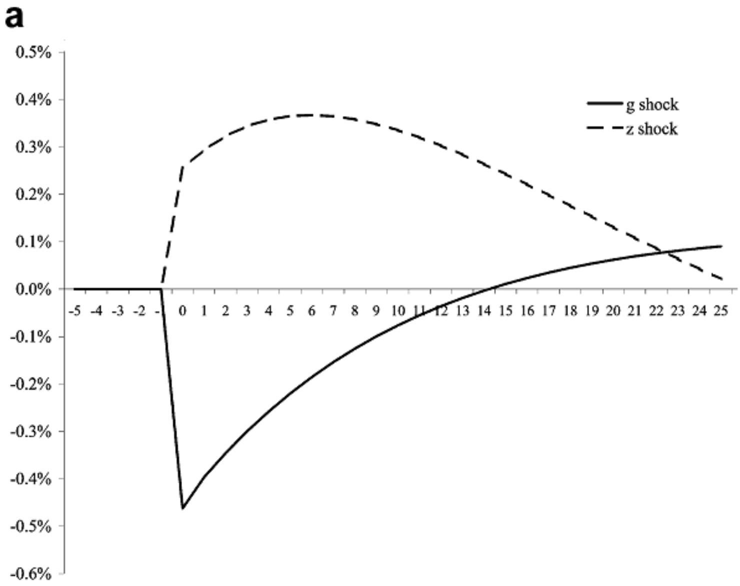
$$\Gamma_t = g_t \Gamma_{t-1} = \prod_{s=0}^{t} g_s$$

- Trend shocks are embedded alongside the standard stationary TFP process A_t:

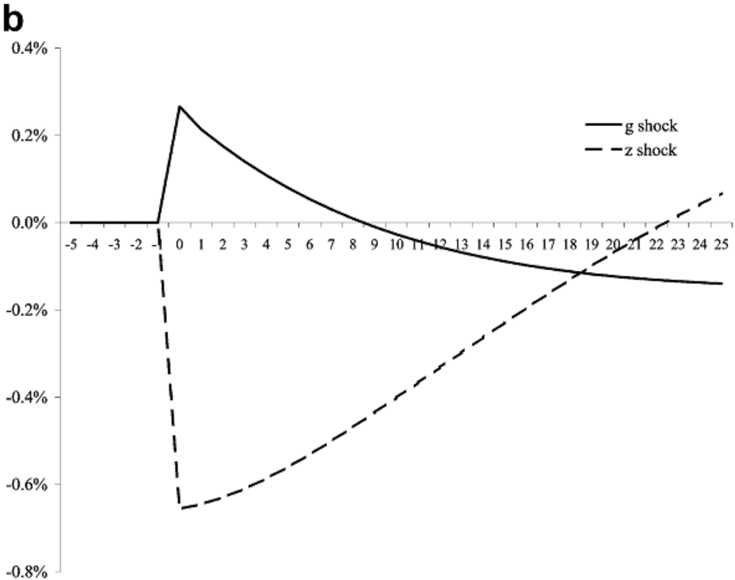
$$\ln A_t = (1-\rho_A) \ln \bar{A} + \rho_A \ln A_{t-1} + \epsilon_{A,t}$$

- Trend shocks dominate in emerging economies (e.g. Mexico), but stationary shocks dominate in advanced economies (e.g. Canada)

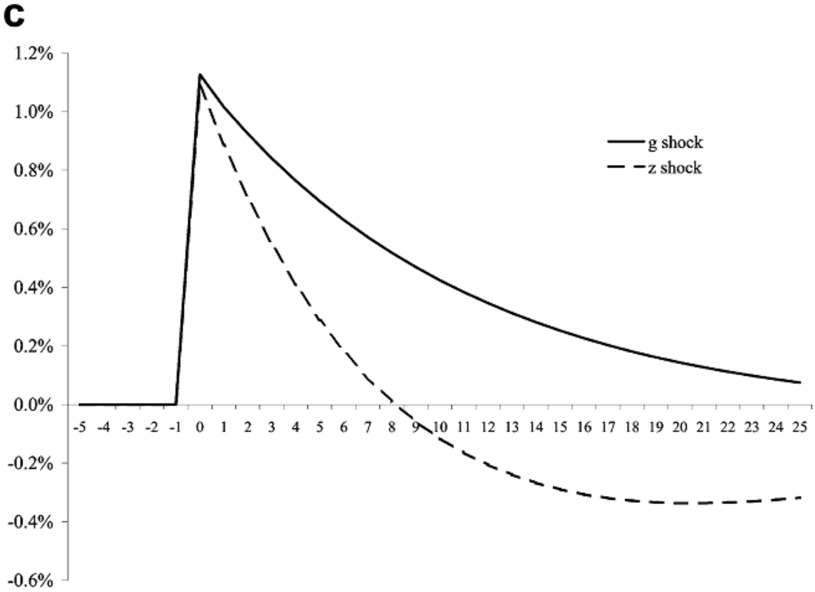
IRF of net exports to GDP



IRF of consumption to GDP



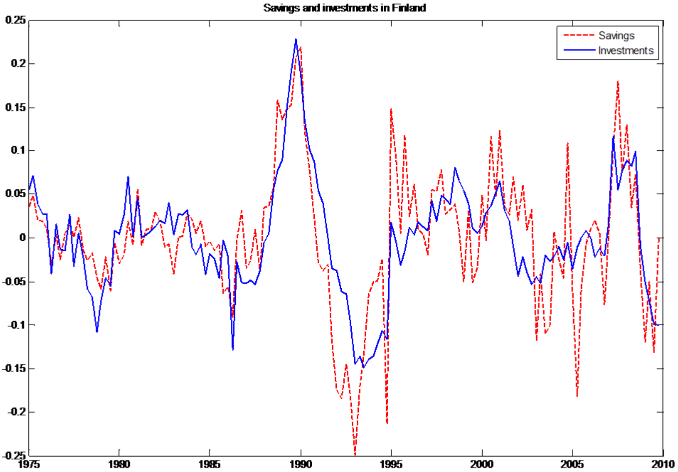
IRF of investment to GDP



Investment and domestic saving in an open economy

- A well-known claim, attributed to Feldstein and Horioka (EJ, 1980) is that under (nearly) integrated international capital markets domestic saving and domestic investment should be uncorrelated.
- The funding of domestic investment does not depend on domestic saving: if you have a good investment project, you can obtain funding from international capital markets
- However, in Finland (and in many other countries) domestic investment and saving are highly correlated over the business cycle.

Investment and domestic saving in Finland



- HP-filtered times series, cyclical component (deviation from the trend)
- Interpretation $0.1 = 10\%$, etc.

Investment and domestic saving in Finland

- Does this mean that Finland is not integrated into international capital markets?
- Not necessarily. The cyclical behavior of investment and saving may derive from a common source (even under integrated capital markets)
- Positive TFP shock:
  - $\Rightarrow$ higher investment
  - but also $\Rightarrow$ higher saving (consumption smoothing motive)

One-good small open economy RBC model: assumptions

- Small open economy produces and consumes a single composite good which is freely traded with the rest of the world.
- Trade of financial assets is limited to a single riskless bond.
- The single tradable type of asset is a consumption-indexed bond which pays a return r_t. This return is exogenously pinned down by $r_t = r_t^*$.
- Representative household maximizes discounted lifetime utility.
- The model features capital adjustment costs.
- Usual transversality condition holds.
- The fact that the interest rate is given from abroad means that the model is not a "general equilibrium" model strictly speaking. Some people prefer to refer to it as a "partial equilibrium" model.

Closing open economy models

- Schmitt-Grohé and Uribe (2003): the small open economy model with incomplete asset markets features:
  - a steady state that depends upon initial conditions
  - equilibrium dynamics that possess a random walk component
- The return on the internationally-traded riskless bond is exogenously determined by $r_t = r_t^*$. In consequence the consumption Euler equation does not pin down the level of consumption in the steady state.
- Steady state of the model depends upon initial conditions and in particular on the net foreign asset position B. This is not desirable, since unconditional variances can't be computed.

Closing open economy models, cont.

- Simplest possible case: no capital, no investment, inelastic labor supply $L_t = 1$, $\rightarrow Y_t = A_t L_t = A_t$ and $W_t = A_t = Y_t$.
- Then (1) can be re-expressed as

$$C_t + B_{t+1} = R_t B_t + W_t \tag{2}$$

- Alternatively, one can define *net foreign debt* $D_{t+1} = -B_{t+1}$.
- But (2) is equivalent to the individual household's budget constraint that we studied in Lecture 1-2 notes.
- Assume that the representative household in the small open economy maximizes $E \sum_{t=0}^{\infty} \beta^t U(C_t)$ subject to (2).
- Then we know (based on results from Lectures 1-2) that the solution to the problem is characterized by the budget constraint (2) and the Euler equation

$$1 = \beta E_t \left[R_{t+1} \frac{U'(C_{t+1})}{U'(C_t)} \right] \tag{3}$$

Closing open economy models, cont.

- Think of finding a steady state. There are two problems with the system (2) and (3).
- *First*, the steady state interest rate $\bar{R} = \bar{R}^* = 1 + \bar{r}^*$ is determined exogenously (in international markets). A steady state where $C_{t+1} = C_t = \bar{C}$ can only exist, if it happens to be so that

$$\bar{R}^* = 1/\beta \tag{4}$$

- *Second,* even if (4) holds, we still have only one equation (2) to pin down two variables B_t and C_t. To close the system, an additional equation is needed.

Consequences for dynamics

- Linearized version of the system becomes

$$\hat{c}_t = E_t \hat{c}_{t+1} \tag{5}$$

and

$$\bar{C}\hat{c}_t + \bar{W}\hat{b}_{t+1} = \bar{R}^* \bar{W}\hat{b}_t + \bar{W}\hat{w}_t \tag{6}$$

where $\hat{b}_t = \frac{B_t - 0}{\bar{W}}$

- From this it follows that

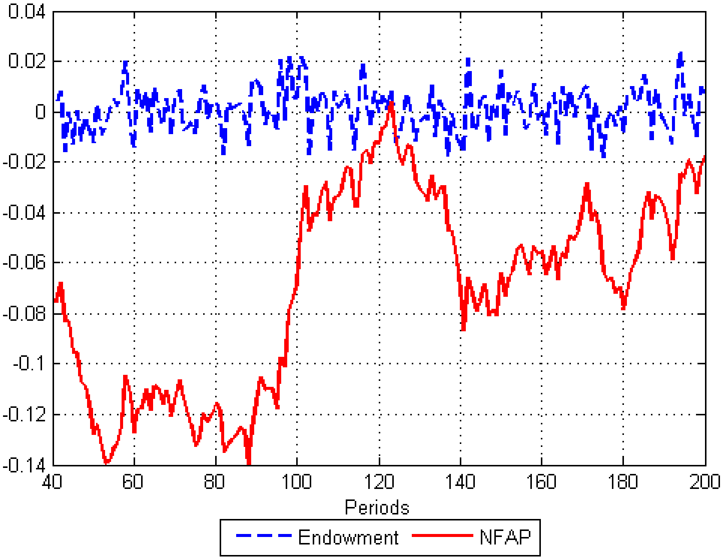
$$\hat{b}_{t+1} = \hat{b}_t + \frac{1}{\bar{R}^*}\hat{w}_t \tag{7}$$

and

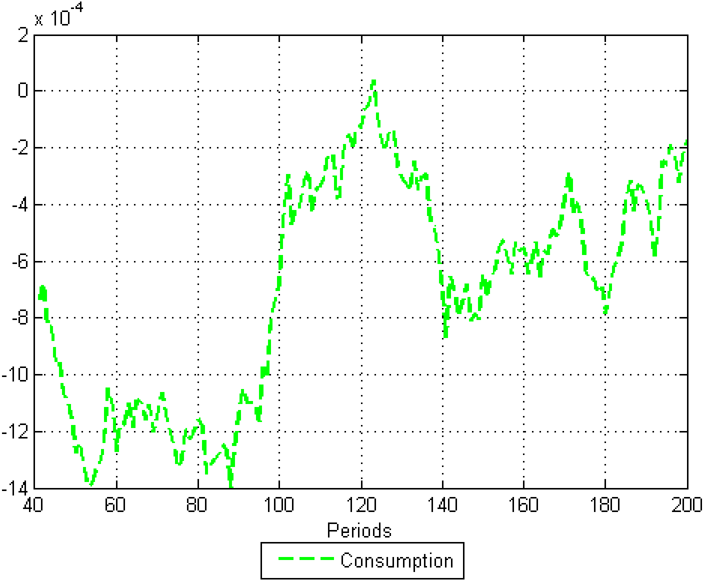
$$\hat{c}_t = \frac{\bar{W}}{\bar{C}}\left(\bar{R}^* - 1\right)\hat{b}_t + \frac{\bar{W}}{\bar{C}\bar{R}^*}\left(\bar{R}^* - 1\right)\hat{w}_t \tag{8}$$

so $\hat{b}_{t+1}$ and $\hat{c}_t$ both follow a unit root process!

Simulation of endowment and NFAP



Simulation of consumption



Closing open economy models: solutions

- In the closed economy model studied in earlier lectures, the solution (or the additional necessary condition) consists of two observations:
- Due to the capital market equilibrium $B_t = K_t$
- The gross rate depends on the capital stock $R_t = \alpha A_t K_t^{\alpha-1} + 1 - \delta$ (assuming that $L_t = 1$)
- Combining, we get $R_t = R(B_t)$ and $R'(B_t) < 0$

Closing open economy models: solutions

- Here we can close the open economy model with a similar assumption: Debt-elastic interest-rate premium:

$$R_t = R(B_t) \equiv R_t^* + \psi(B_t),\ \psi'(B_t) < 0 \tag{9}$$

- Assume that $B_t < 0$, and there is net foreign debt. If net foreign debt $D_t = -B_t$ increases (B_t goes down), borrowing gets more expensive.
- If $B_t > 0$ and the positive net asset position becomes larger, it is harder to find profitable investment opportunities abroad.
- The premium can be treated as a proxy for country riskiness relative to the rest of the world.
- Now the system (2), (3) and (9) constitutes a complete model: three equations for three variables ($B_t,\ C_t$ and R_t).
- It is straightforward to augment this model with investment and capital accumulation in the home economy.

Closing open economy models: solutions

- A number of other techniques have also been proposed to close small open economy models (see Schmitt-Grohé and Uribe, JIE 2003)
- Endogenous discount factor θ (Uzawa (1968), Mendoza (1991)):

$$\theta_{t+1} = \beta(C, L)\theta_t, \ \beta_C < 0$$

- Convex portfolio adjustment costs:

$$B_{t+1} = R_t B_t + Y_t - C_t - I_t - \Omega(I_t, K_t) - \psi(B_t - \bar{B})^2$$

- Complete asset markets: Agents have access to complete array of a state-contingent claims. Here consumption Euler equation holds every period, not just in expectations.

$$U_c(C_t, L_t) = \beta R_{t+1} U_c(C_{t+1}, L_{t+1})$$

- As discussed by Schmitt-Grohé and Uribe (2003), the first three different ways of closing open economy models yield very similar behavior in terms of (conditional) second moments and impulse responses. Only the model with complete asset markets behaves somewhat differently.

Small open economy RBC model with debt-elastic interest rate premium and capital adjustment costs

$$\max_{\{C_t, L_t, I_t, K_{t+1}, B_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, L_t\right)$$

$$st.$$

$$B_{t+1} = R_t B_t + A_t K_t^{\alpha} L_t^{1-\alpha} - C_t - I_t - \Omega\left(I_t, K_t\right) \quad (10)$$

$$K_{t+1} = \left(1-\delta\right) K_t + I_t \quad (11)$$

$$R_t = R_t^* + \psi\left(\widetilde{B}_t\right) \quad (12)$$

$$\ln(A_t) = \left(1-\rho\right) \ln \bar{A} + \rho \ln A_{t-1} + \varepsilon_t \quad (13)$$

- Note: K is domestic capital stock and B is (net) domestic holdings of foreign assets
- Interest rate premium depends on average asset holdings in the economy $\widetilde{B}_t$ ($\widetilde{B}_t$ is exogenous to the representative household)
- Foreign interest rate R_t^* exogenous

Small open economy RBC model, cont.

- Bellman equation

$$
\begin{aligned}
& V\left(B_t, K_t; A_t, \widetilde{B}_t, R_t^*\right) \\
= \quad & \max_{B_{t+1}, K_{t+1}, L_t} U\left(C_t, L_t\right) + \beta E_t\left[V\left(B_{t+1}, K_{t+1}; A_t, \widetilde{B}_t, R_{t+1}^*\right)\right] \\
& st. \\
C_t = \quad & A_t K_t^{\alpha} L_t^{1-\alpha} + R_t B_t - B_{t+1} \\
& + (1-\delta) K_t - K_{t+1} - \Omega\left(K_{t+1} - (1-\delta) K_t, K_t\right) \\
R_t = \quad & R_t^* + \psi\left(\widetilde{B}_t\right)
\end{aligned}
$$

- Endogenous state variables B_t, K_t
- Exogenous state variables $A_t, \widetilde{B}_t, R_t^*$

Small open economy RBC model, cont.

- First-order conditions:
  - with respect to B_{t+1}

$$U_C\left(C_t, L_t\right) = E_t\left[V_B\left(B_{t+1}, K_{t+1}; A_t, \widetilde{B}_t, R^*_{t+1}\right)\right]$$

  - with respect to K_{t+1}

$$U_C\left(C_t, L_t\right)\left(1+\Omega_I\left(I_t, K_t\right)\right) = E_t\left[V_K\left(B_{t+1}, K_{t+1}; A_t, \widetilde{B}_t, R^*_{t+1}\right)\right]$$

  - with respect to L_t

$$-U_L\left(C_t, L_t\right) = \left(1-\alpha\right) A_t\left(\frac{K_t}{L_t}\right)^{\alpha} U_C\left(C_t, L_t\right)$$

Small open economy RBC model, cont.

- Envelope conditions (indirect effects can be ignored!)

$$V_B\left(B_t, K_t; A_t, \widetilde{B}_t, R_t^*\right) = R_t U_C\left(C_t, L_t\right)$$

$$V_K\left(B_t, K_t; A_t, \widetilde{B}_t, R_t^*\right) =$$
$$U_C\left(C_t, L_t\right)\left(\alpha A_t\left(\frac{L_t}{K_t}\right)^{1-\alpha} + (1-\delta)\left(1+\Omega_I\left(I_t, K_t\right)\right) - \Omega_K\left(I_t, K_t\right)\right)$$

- Lead the envelope conditions by one period, take expectations and plug into the first-order conditions $\Rightarrow$ Euler equations

Dynamic equilibrium

- Two Euler equations (notice that in equilibrium $\widetilde{B}_{t+1} = B_{t+1}$)

$$1 = \beta E_t \left[\frac{U_C\left(C_{t+1}, L_{t+1}\right)}{U_C\left(C_t, L_t\right)} \left(R^*_{t+1} + \psi\left(B_{t+1}\right)\right) \right]$$

$$1 + \Omega_I\left(I_t, K_t\right) = \beta E_t \left[\tfrac{U_C\left(C_{t+1}, L_{t+1}\right)}{U_C\left(C_t, L_t\right)} \left(\alpha A_{t+1} \left(\tfrac{L_{t+1}}{K_t + 1} \right)^{1-\alpha} \right. \right. \\ \left. \left. + \left(1-\delta\right)\left(1 + \Omega_I\left(I_{t+1}, K_{t+1}\right)\right) - \Omega_K\left(I_{t+1}, K_{t+1}\right)\right)\right]$$

- Labor market condition

$$-U_L\left(C_t, L_t\right) = \left(1-\alpha\right) A_t \left(\frac{K_t}{L_t}\right)^{\alpha} U_C\left(C_t, L_t\right)$$

- Budget constraint

$$\begin{aligned} C_t &= A_t K_t^{\alpha} L_t^{1-\alpha} + R_t B_t - B_{t+1} \\ &\quad + \left(1-\delta\right) K_t - K_{t+1} - \Omega\left(K_{t+1} - \left(1-\delta\right) K_t, K_t\right) \end{aligned}$$

Dynamic equilibrium

- Four variables (C, L, K, B), four equations
- $+$ the exogenous law of motion of TFP A_t
- $+$ an exogenous process for foreign interest rate R_t^*

Functional forms

- Following Schmitt-Grohé and Uribe (2003) we assume the following functional forms
  - utility
$$U\left(C_t, L_t\right) = \frac{\left(C_t - \theta^{-1} L_t^{\theta}\right)^{1-\sigma}}{1-\sigma}$$
  - capital adjustment costs
$$\Omega(I_t, K_t) = \frac{\kappa}{2}\left(I_t - \delta K_t\right)^2 = \frac{\kappa}{2}\left(K_{t+1} - K_t\right)^2$$
  - risk premium
$$\psi\left(\widetilde{B}_t\right) = \psi\left(e^{\bar{B} - \widetilde{B}_t} - 1\right)$$
- We also assume that
$$R_t^* = \bar{R}^* = 1/\beta$$

GHH preferences

- Standard preferences deliver too smooth consumption relative to that observed in advanced economies. In emerging markets consumption is still more volatile.
- Greenwood, Hercowitz, Huffman (AER, 1988) proposed preferences with income effect killed:

$$U\left(C_t, L_t\right) = \frac{\left(C_t - \frac{L_t^{\theta}}{\theta}\right)^{1-\sigma}}{1-\sigma} \tag{14}$$

- Problem: these preferences are not consistent with the balanced growth path, as formalized in King, Plosser and Rebelo (JME, 1988)
- Solution: adding trend to disutility of labor, proposed by Aguiar and Gopinath (JPE, 2007, NBER 2005 version):

$$U\left(\tilde{C}_t, L_t\right) = \frac{\left(\tilde{C}_t - \tilde{X}_t \frac{L_t^{\theta}}{\theta}\right)^{1-\sigma}}{1-\sigma} \tag{15}$$

Financial market frictions

Economic activities that require financing typically involve cooperation/interaction between the suppliers of funds and the users of funds.

Conflicts can arise in this relationship due to asymmetric information:

- The lender does not know / trust the borrower
  - "lemons" problem: how can the lender distinguish between good firms and bad ones
- The outcome of almost all economic activities involves some uncertainty, and the people undertaking the activity usually know best what the nature of the uncertainty is.
- This can create a conflict, since the borrower (user of funds) has an incentive to report that things did not go well (bad luck). Why?
  - the borrower wants to pay just a little to the lender and keep the lion's share himself, or
  - the borrower just wants to apply low effort to the project

External finance premium

- Due to financial frictions / asymmetric information there is an external finance premium
  - entrepreneurs have to pay for external funds more than the market interest rate
- The size of the external finance premium depends on the shape of the firm's balance sheet
  - if the firm has a high assets-to-equity ratio, it has to pay a higher price for external funds

Financial frictions

- Bernanke and Gertler (1989), workhorse model of financial frictions
  - BG is currently a building block of many macro models, which try to address financial frictions
- Carlstrom and Fuerst (1997) embed BG into a standard RBC model.
- Some well-known papers using the BG approach:
  - Bernanke, Gertler and Gilchrist (1999): The financial accelerator in quantitative business cycle framework. *Handbook of Macroeconomics, Chapter 21*
  - Christiano, Motto and Rostagno (2014): Risk shocks. *AER*

Alternative ways to model financial frictions

Collateral constraints

- You cannot borrow more than the value of your collateral (the collateral can be e.g. your house)
- Kiyotaki and Moore (1997): Credit cycles. *JPE*
  Iacoviello (2005): House prices, borrowing constraints and monetary policy in the business cycle. *AER*

Moral hazard

- Holmström and Tirole (1997): Financial intermediation, loanable funds, and the real sector. *QJE*
- A tractable framework, where both firm balance sheets and bank balance sheets can be analyzed simultaneously
- Entrepreneurs face incentives to choose socially suboptimal project, bankers monitor.

Key features of the BG approach

To model lending relationships and asymmetric information, we need to depart from the representative household assumption. In BG there are two kinds of agents:

- *Entrepreneurs* have good ideas (well, at least some of them have) but not enough funds
- Ordinary *households* have extra funds but no good ideas

Households lend funds to entrepreneurs.

At the aggregate level the value of entrepreneurs' net worth (equity) constrains the feasible level of investment through the endogenous *risk premium* (credit spread).

A shock that destroys entrepreneurs' net worth lowers investment and economic growth at the aggregate level.

Net worth, investment and leverage

- Denote an entrepreneur's own funds, or his "net worth", by a_t
  - Note: entrepreneurs differ in terms of a_t; some entrepreneurs have higher net worth than others.
- If an entrepreneur with net worth a_t wants to make an investment i_t^a, which exceeds his net worth (i.e. $i_t^a > a_t$), he must borrow the amount

$$i_t^a - a_t$$

- The entrepreneur's leverage (assets-to-equity ratio) is given by

$$i_t^a / a_t$$

Idiosyncratic shocks

- In each period, every entrepreneur draws an idiosyncratic shock ω from a cumulative distribution function $\Phi(\omega)$. We also assume that $E\omega = 1$ (a useful normalization).
- The random variable ω is realized *after* the entrepreneur has invested i_t^a.
- An entrepreneur, who invests i_t^a, produces

$$\omega i_t^a$$

  units of a capital good.
- The capital good is sold at market price q_t so that the consumption good value of ωi_t^a is $q_t \omega i_t^a$

Asymmetric information and costly state verification

- After ω is realized, only the entrepreneur knows its value.
- An outsider must pay a monitoring cost to observe the realization ω.
  - This is called the costly state verification problem.

Why an equity-type contract is not efficient

- Suppose there was an equity-type contract stipulating that the entrepreneur must pay a certain fraction γ of the proceeds ωi_t^a to the lender.
- Then the entrepreneur would face incentives to underreport the realization of ω.
- The entrepreneur tells the lender that revenues amount to $\omega_{\min} i_t^a$ (where $\omega_{\min}$ is the minimum value of ω) pays to the lender the amount $\gamma\omega_{\min} i_t^a$ and keeps the remaining revenues $(\omega - \gamma\omega_{\min})\, i_t^a$.
- If the lender wants to avoid being duped, he has to constantly monitor the entrepreneur. Since monitoring is costly, this is not an efficient arrangement.
- Simple standard debt contract works better than an equity-type contract in this costly state verification environment.
- In equilibrium, there is monitoring only when the true realization ω is low and the entrepreneur cannot pay back the debt.

Debt contract

The debt contract is of the following form:

- An entrepreneur, who borrows $\left(i_t^a - a_t\right)$ must pay back

$$R_t^a \left(i_t^a - a_t\right)$$

  (units of consumption good) where R_t^a is the gross interest rate specified in the debt contract.
- If the realization of the shock ω is so low that the entrepreneur cannot pay $R_t^a \left(i_t^a - a_t\right)$, he must repay the bank whatever he has, i.e. ωi_t^a. In other words, the entrepreneur declares bankruptcy.
- When the entrepreneur declares bankruptcy, the lender verifies the state ω.
- If the lender didn't monitor, the entrepreneur would just report $\omega = \omega_{\min}$ and pay the lender $\omega_{\min} i_t^a$, while keeping the rest to himself.
- The lender must spend $\mu q_t i_t^a$ units of final goods to monitor the (defaulting) entrepreneur.

Debt contract, cont.

- There is a value of ω, $\bar{\omega}_t^a$, such that for all $\omega < \bar{\omega}_t^a$ it is infeasible for the entrepreneur with net worth a_t to repay the debt. This threshold value $\bar{\omega}_t^a$ satisfies

$$R_t^a \left(i_t^a - a_t\right) - q_t \bar{\omega}_t^a i_t^a = 0 \tag{16}$$

- The debt contract can be specified in terms of $\left(R_t^a, i_t^a\right)$, or equivalently in terms of $\left(\bar{\omega}_t^a, i_t^a\right)$.
- For a given i_t^a, (16) defines a one-to-one mapping between R_t^a and $\bar{\omega}_t^a$:

$$R_t^a = q_t \bar{\omega}_t^a \left(\frac{i_t^a}{i_t^a - a_t}\right) \tag{17}$$

Entrepreneurs' profits

- Average profits of across all entrepreneurs with net worth a_t, who invest i_t^a is

$$
\begin{aligned}
& \underbrace{q_t i_t^a \int_0^\infty \omega d\Phi(\omega)}_{\text{average revenues}} - \underbrace{\int_{\bar{\omega}_t^a}^\infty R_t^a \left(i_t^a - a_t\right) d\Phi(\omega)}_{\text{average costs of non-bankrupt entrepreneurs}} \\
& - \underbrace{q_t i_t^a \int_0^{\bar{\omega}_t^a} \omega d\Phi(\omega)}_{\text{average costs of bankrupt entrepreneurs}} \\
= \; & q_t i_t^a \int_0^\infty \omega d\Phi(\omega) - \int_{\bar{\omega}_t^a}^\infty q_t \bar{\omega}_t^a i_t^a d\Phi(\omega) - q_t i_t^a \int_0^{\bar{\omega}_t^a} \omega d\Phi(\omega) \\
= \; & q_t i_t^a \left[\int_{\bar{\omega}_t^a}^\infty \omega d\Phi(\omega) - \bar{\omega}_t^a \left(1 - \Phi\left(\bar{\omega}_t^a\right)\right)\right] = q_t i_t^a f\left(\bar{\omega}_t^a\right)
\end{aligned}
$$

Earnings of the lenders

$$\begin{aligned}
& \underbrace{q_t i_t^a \left[\int_0^{\bar{\omega}_t^a} \omega d\Phi(\omega) - \mu \Phi(\bar{\omega}_t^a) \right]}_{\text{income from bankrupt entrepreneurs}} + \underbrace{R_t^a \left(i_t^a - a_t \right) \left[1 - \Phi(\bar{\omega}_t^a) \right]}_{\text{income from non-bankrupt entrepreneurs}} \\
= \quad & q_t i_t^a \left[\int_0^{\bar{\omega}_t^a} \omega d\Phi(\omega) - \mu \Phi(\bar{\omega}_t^a) \right] + q_t \bar{\omega}_t^a i_t^a \left[1 - \Phi(\bar{\omega}_t^a) \right] \\
= \quad & q_t i_t^a \left[\underbrace{\int_0^{\bar{\omega}_t^a} \omega d\Phi(\omega) - \mu \Phi(\bar{\omega}_t^a)}_{\text{from bankrupt}} + \underbrace{\bar{\omega}_t^a \left(1 - \Phi(\bar{\omega}_t^a) \right)}_{\text{from non-bankrupt}} \right] \\
= \quad & q_t i_t^a g(\bar{\omega}_t^a)
\end{aligned}$$

Entrepreneurs' share of proceeds

- To recap, entrepreneurs get the fraction

$$
\begin{aligned}
f\left(\bar{\omega}_t^a\right) &= \int_{\bar{\omega}_t^a}^{\infty} \omega d\Phi\left(\omega\right) - \bar{\omega}_t^a\left(1-\Phi\left(\bar{\omega}_t^a\right)\right) \\
&= \underbrace{\left(1-\Phi\left(\bar{\omega}_t^a\right)\right)}_{\text{Pr(not bankrupt)}} \times \\
&\qquad \underbrace{E_\omega\left[\left(\omega-\bar{\omega}_t^a\right) \mid \omega \geq \bar{\omega}_t^a\right]}_{\text{non-bankrupt entrepreneurs' expected share}}
\end{aligned}
$$

Lenders' share of proceeds

- Lenders get the fraction

$$
\begin{aligned}
g\left(\bar{\omega}_t^a\right) &= \left[\underbrace{\int_0^{\bar{\omega}_t^a} \omega d\Phi\left(\omega\right) - \mu\Phi\left(\bar{\omega}_t^a\right)}_{\text{from bankrupt}} + \underbrace{\bar{\omega}_t^a\left(1-\Phi\left(\bar{\omega}_t^a\right)\right)}_{\text{from non-bankrupt}}\right] \\
&= \underbrace{\Phi\left(\bar{\omega}_t^a\right)}_{\Pr(\text{bankrupt})} \times \underbrace{\left\{E\left[\left(\omega-\mu\right) \mid \omega < \bar{\omega}_t^a\right]\right\}}_{\text{all revenues, net of bankruptcy costs}} \\
&+ \underbrace{\left(1-\Phi\left(\bar{\omega}_t^a\right)\right)}_{\Pr(\text{not bankrupt})} \times \underbrace{\bar{\omega}_t^a}_{\text{contractual lenders' share}}
\end{aligned}
$$

- The sum of the two fractions is

$$
\begin{aligned}
& f\left(\bar{\omega}_t^a\right)+g\left(\bar{\omega}_t^a\right) \\
= \quad & \underbrace{\int_{\bar{\omega}_t^a}^{\infty} \omega d\Phi\left(\omega\right)-\bar{\omega}_t^a\left(1-\Phi\left(\bar{\omega}_t^a\right)\right)}_{f\left(\bar{\omega}_t^a\right)} \\
& +\underbrace{\int_0^{\bar{\omega}_t^a} \omega d\Phi\left(\omega\right)-\mu\Phi\left(\bar{\omega}_t^a\right)+\bar{\omega}_t^a\left(1-\Phi\left(\bar{\omega}_t^a\right)\right)}_{g\left(\bar{\omega}_t^a\right)} \\
= \quad & E\left[\omega\right]-\mu\Phi\left(\bar{\omega}_t^a\right)=1-\mu\Phi\left(\bar{\omega}_t^a\right)
\end{aligned}
$$

  so that on average the share $\mu\Phi\left(\bar{\omega}_t^a\right)$ of produced capital is destroyed in monitoring.

  - μ is the cost of bankruptcy
  - $\Phi\left(\bar{\omega}_t^a\right)$ is the probability of bankruptcy
  - Remember the normalization $E\left[\omega\right]=1$

Equilibrium debt contract

- For simplicity of exposition, suppose that the loan period is *intra-temporal*, so that the opportunity cost of lending for the lender is a (safe) zero interest rate.
- The lenders are also *risk-neutral*.
- Thus the lenders face the following constraint when lending to entrepreneurs:

$$q_t i_t^a g\left(\bar{\omega}_t^a\right) \geq i_t^a - a_t \tag{18}$$

- The constraint (18) tells that the lenders have to earn a non-negative return from the loans extended to entrepreneurs.

Equilibrium debt contract, cont.

- Competition between lenders ensures that, in equilibrium, they make zero profits, i.e. the constraint is binding.
- Then the equilibrium debt contract maximizes entrepreneurs' pay-offs subject to the non-negativity constraint (18).
- The problem of maximizing the entrepreneur's expected pay-offs subject to the lenders' zero profit condition has the following Lagrangian presentation

$$\max_{\bar{\omega}_t^a, i_t^a} \quad \underbrace{q_t i_t^a f\left(\bar{\omega}_t^a\right)}_{\text{entrepreneur's expected payoff}} + \lambda^a \underbrace{\left[q_t i_t^a g\left(\bar{\omega}_t^a\right) - i_t^a + a_t\right]}_{\text{lender's zero profit condition}} \tag{19}$$

- The first order conditions of this problem are

$$q_t f\left(\bar{\omega}_t^a\right) + \lambda^a \left[q_t g\left(\bar{\omega}_t^a\right) - 1\right] = 0 \quad (\text{foc wrt } i_t^a) \tag{20}$$

$$q_t i_t^a f'\left(\bar{\omega}_t^a\right) + \lambda^a q_t i_t^a g'\left(\bar{\omega}_t^a\right) = 0 \quad (\text{foc wrt } \bar{\omega}_t^a) \tag{21}$$

$$q_t i_t^a g\left(\bar{\omega}_t^a\right) - i_t^a + a_t = 0 \tag{22}$$

The entrepreneur's trade-off

- Essentially, the entrepreneur faces the following trade-off:
- He/she would like to raise more funds from outside lenders and increase leverage i_t / a_t because larger investment scale means higher income
- However, getting more funds from outsiders also implies a higher interest rate R_t^a (see eq. (17))
- or, equivalently, promising a bigger expected share of proceeds $g\left(\bar{\omega}_t^a\right)$ to the lenders,
- so that the entrepreneur himself/herself will get a smaller share $f\left(\bar{\omega}_t^a\right)$.
- Also, the higher the leverage, the higher the probability of bankruptcy,
- which is costly (due to state verification costs).
- On the other hand, settling for lower leverage i_t / a_t means a lower scale of investment, but a also larger share of proceeds for the entrepreneur, and bankruptcy is less likely.

Equilibrium debt contract, cont.

- Combine equations (20) and (21) to substitute out λ^a

$$\begin{aligned} q_t f\left(\bar{\omega}_t^a\right) &= \frac{f'\left(\bar{\omega}_t^a\right)}{g'\left(\bar{\omega}_t^a\right)}\left[q_t g\left(\bar{\omega}_t^a\right) - 1\right] && (23) \\ q_t i_t^a g\left(\bar{\omega}_t^a\right) &= i_t^a - a_t && (24) \end{aligned}$$

- Note from (23) that $\bar{\omega}_t^a$ is a function of q_t (the price of capital) only
  - $\bar{\omega}_t^a$ does not depend on the net worth of the entrepreneur a_t
  - Thus we can drop the superscript a from $\bar{\omega}_t$
- Also, from (24) one can see that the ratio i_t / a_t is independent from the entrepreneur's net worth.
- Hence, all entrepreneurs choose the same leverage ratio i_t / a_t.

Equilibrium leverage

- Equilibrium leverage can then be solved from equation (24)

$$\frac{i_t}{a_t} = \left[1 - q_t g\left(\bar{\omega}_t^a\right)\right]^{-1} \tag{25}$$

- Notice: The higher the leverage, the higher the lenders' share, $g\left(\bar{\omega}_t\right)$.
- Also, the higher the price of capital q_t, the higher the leverage.

Bankruptcy threshold

- Applying the Leibnitz rule, we get

$$\begin{aligned} f'(\bar{\omega}_t) &= -(1-\Phi(\bar{\omega}_t)) \\ g'(\bar{\omega}_t) &= (1-\Phi(\bar{\omega}_t)) - \mu\Phi'(\bar{\omega}_t) \end{aligned}$$

- Then the first order condition (23) boils down to

$$\underbrace{\left(\frac{q_t f(\bar{\omega}_t)}{1-q_t g(\bar{\omega}_t)}\right)}_{\text{return to entrepreneurial capital } a_t} = \frac{1}{1-\mu h(\bar{\omega}_t)} \tag{26}$$

  where

$$h(\bar{\omega}_t) = \frac{\Phi'(\bar{\omega}_t)}{1-\Phi(\bar{\omega}_t)}$$

  is the (so called) hazard ratio of bankruptcy: the increase in the relative probability of bankruptcy, if $\bar{\omega}_t$ is increased (or equivalently, if the interest rate R_t is raised).

- The bankruptcy threshold $\bar{\omega}_t$ is determined by the equation (26)

- The first-order condition (26) can be further re-written as

$$\begin{aligned}
q_t \underbrace{\left(f\left(\bar{\omega}_t\right) + g\left(\bar{\omega}_t\right)\right)}_{=1-\mu\Phi(\bar{\omega}_t)} &= 1 + \mu q_t h\left(\bar{\omega}_t\right) f\left(\bar{\omega}_t\right) \\
\mu\left[\Phi\left(\bar{\omega}_t\right) + h\left(\bar{\omega}_t\right) f\left(\bar{\omega}_t\right)\right] &= \frac{q_t - 1}{q_t} \\
\mu\left[\Phi\left(\bar{\omega}_t\right) + h\left(\bar{\omega}_t\right)\int_{\bar{\omega}_t}^{\infty}\left(\omega - \bar{\omega}_t\right) d\Phi\left(\bar{\omega}_t\right)\right] &= \frac{q_t - 1}{q_t}
\end{aligned}$$

- Notice, without bankruptcy costs μ, there would be no financial frictions in this model.
  - Then eq. (26) would imply the equilibrium return to entrepreneurial capital would be 1 ($=$ the intra-period rate of return to households' investment).
  - Also the price of capital q_t would be 1.

Equilibrium contract, redux

- Here's an alternative way to solve for the equilibrium debt contract:
  1. Solve for leverage from the lenders' zero profit condition

     $$\frac{i_t}{a_t} = \left[1 - q_t g\left(\bar{\omega}_t\right)\right]^{-1} \Leftrightarrow i_t = \frac{a_t}{1 - q_t g\left(\bar{\omega}_t\right)}$$

  2. Plug leverage into the entrepreneurs' objective function. We get an unconstrained maximization problem

     $$\max_{\bar{\omega}_t} \left(\frac{f\left(\bar{\omega}_t\right)}{1 - q_t g\left(\bar{\omega}_t\right)}\right) q_t a_t \tag{27}$$

     From this maximization problem, it is easy to see, that the entrepreneur faces the trade-off between:
     i) getting him/herself a large fraction $f\left(\bar{\omega}_t\right)$ of (exptected) revenues, and
     ii) large investment scale and high leverage $\left[1 - q_t g\left(\bar{\omega}_t\right)\right]^{-1}$, which would call for a large fraction $g\left(\bar{\omega}_t\right)$ of (expected) revenue being paid to lenders.

Risk premium 1

- From (17) we can deduce that entrepreneurs pay the same interest rate, regardless of their net worth level a_t:

$$R_t = q_t \bar{\omega}_t \left(\frac{i_t}{i_t - a_t} \right) = \frac{q_t \bar{\omega}_t}{1 - a_t / i_t} = \frac{\bar{\omega}_t}{g\left(\bar{\omega}_t\right)}$$

  where the last expression $\frac{\bar{\omega}_t}{g(\bar{\omega}_t)}$ follows from (25).

- Lending to an individual entrepreneur is risky. The entrepreneur may declare bankruptcy, in which case the lender does not get back the promised amount and has to incur the monitoring cost on top of that.
- The lending rate is increasing in the default threshold $\bar{\omega}_t$ (i.e. probability of default).
- The risk premium the lender charges is the excess of R_t over the sure rate of return (which in this case is 1):

$$R_t - 1 = \frac{\bar{\omega}_t}{g\left(\bar{\omega}_t\right)} - 1$$

Risk premium 2

- An entrepreneur who borrows $i_t - a_t$, pays back the expected sum $(1 - f(\bar{\omega}_t))\, q_t i_t$.
- Hence the expected cost of funding is $\frac{(1-f(\bar{\omega}_t))i_t}{i_t - a_t}$, and we get another measure of the risk premium

$$\rho_t = \frac{(1 - f(\bar{\omega}_t))\, q_t i_t}{i_t - a_t} - 1 = \frac{(1 - f(\bar{\omega}_t))\, q_t}{1 - a_t / i_t} - 1$$

- Next, since $a_t / i_t = 1 - q_t g(\bar{\omega}_t)$

$$\rho_t = \frac{q_t (1 - f(\bar{\omega}_t) - g(\bar{\omega}_t))}{q_t g(\bar{\omega}_t)}$$

Risk premium 3

- Finally, since $1 - f(\bar{\omega}_t) - g(\bar{\omega}_t) = \mu\Phi(\bar{\omega}_t)$, we get

$$\rho_t = \frac{\mu\Phi(\bar{\omega}_t)}{g(\bar{\omega}_t)}$$

- Hence, this measure essentially links the risk premium to expected bankruptcy costs. The risk premium is the ratio of bankruptcy costs $(\mu\Phi(\bar{\omega}_t))$ to lenders' share (proceeds) $g(\bar{\omega}_t)$
-

$$\begin{aligned}
\rho_t'(\bar{\omega}_t) &= \frac{\mu\Phi'(\bar{\omega}_t)}{g(\bar{\omega}_t)} - \rho_t \frac{g'(\bar{\omega}_t)}{g(\bar{\omega}_t)} \\
&= \frac{[-\rho_t(1-\Phi(\bar{\omega}_t)) + (1+\rho_t)\mu\Phi'(\bar{\omega}_t)]}{g(\bar{\omega}_t)} \\
&= \frac{(1-\Phi(\bar{\omega}_t))}{g(\bar{\omega}_t)}\left[(1+\rho_t)\mu h(\bar{\omega}_t) - \rho_t\right]
\end{aligned}$$

Equilibrium debt contract, cont.

- An entrepreneur with net wealth a_t invests

$$i_t = \frac{1}{1 - q_t g\left(\bar{\omega}_t\right)} a_t$$

where the term $\frac{1}{1-q_t g(\bar{\omega}_t)}$ captures leverage

- The entrepreneur's (ex ante) net revenues are

$$q_t f\left(\bar{\omega}\right) i_t = \frac{q_t f\left(\bar{\omega}\right)}{1 - q_t g\left(\bar{\omega}_t\right)} a_t$$

Aggregation

- Since investment depends linearly on the entrepreneur's net worth, aggregation is very easy. Aggregate investment I_t is given by

$$I_t = \frac{1}{1 - q_t g\left(\bar{\omega}_t\right)} A_t$$

  where A_t is the net worth of all entrepreneurs in the economy.
- Entrepreneurs' aggregate net revenue is

$$NR = q_t f\left(\bar{\omega}\right) I_t = \frac{q_t f\left(\bar{\omega}_t\right)}{1 - q_t g\left(\bar{\omega}_t\right)} A_t$$

- Aggregate capital accumulation is

$$\begin{aligned} K_{t+1} &= \left(1-\delta\right) K_t + I_t \left(1 - \mu \Phi\left(\bar{\omega}_t\right)\right) \\ &= \left(1-\delta\right) K_t + \left(\frac{1 - \mu \Phi\left(\bar{\omega}_t\right)}{1 - q_t g\left(\bar{\omega}_t\right)}\right) A_t \end{aligned}$$

- Importantly, aggregate investment is constrained by entrepreneurs' net worth.