Equity premium & price of capital

Macroeconomic Theory, Lectures 7-8

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Course material

Readings for lectures 7-8:

- DeJong and Dave (2011), Ch.3.3
- Wickens (2011), Ch.10, Ch.11.1-11.2
- Mehra and Prescott (JME, 1985), "The Equity Premium: A Puzzle"
- Romer (2012), Ch.9

Choice over risky assets and equity premium

- As of now, our economy has had only one asset (B_t), which pays out a return R_t. More generally, households could make portfolio choices over several (risky) assets.
- Next, we consider a problem in which household can choose between two assets. One is risky and one is riskless.
- Household's solution to the portfolio choice problem between two assets gives us an arbitrage condition between the returns of different assets.
- Using the arbitrage condition, we will compute the equity premium. We will find out that our model generates too small equity premium ("Equity Premium Puzzle").

The problem

- Representative consumer chooses consumption stream $\{C_{t+i}\}_{i=0}^{\infty}$ in order to:

$$
\begin{aligned}
\max E_t \sum_{i=0}^{\infty} \beta^i U(C_{t+i}),\ \beta &\in (0,1) \quad s.t. \\
S_{t+1} + B_{t+1} &= R_t^S S_t + R_t^B B_t + W_t - C_t \\
\lim_{i\to\infty} E_t \frac{S_{t+i}}{\Pi_{j=1}^{i} R_{t+j}^S} &= 0,\ \lim_{i\to\infty} E_t \frac{B_{t+i}}{\Pi_{j=1}^{i} R_{t+j}^B} = 0
\end{aligned}
$$

- State variables, and thus predetermined variables are S_t and B_t.
- Assume that returns R_t^S and R_t^B follow first order Markov process, which are exogenous to the consumer. R_{t+1}^B is known already in period t.
- Think of S_t as stock of equities of (listed) firms, B_t is the riskless asset ("bond").

Bellman equation:

$$
\begin{aligned}
V(S_t, B_t; R_t^S, R_t^B) &= \max_{C_t, S_{t+1}, B_{t+1}} \{U(C_t) \qquad (1)\\
&\quad +\beta E_t \left[V(S_{t+1}, B_{t+1}; R_{t+1}^S, R_{t+1}^B) \right] \} \\
&\quad s.t. \\
S_{t+1} + B_{t+1} &= R_t^S S_t + R_t^B B_t + W_t - C_t \qquad (2)
\end{aligned}
$$

- Turn this into unconstrained problem by plugging (2) into Bellman equation

$$
V(S_t, B_t; R_t^S, R_t^B) = \max_{S_{t+1}, B_{t+1}} \left\{ \begin{array}{c} U(R_t^S S_t + R_t^B B_t + W_t - S_{t+1} - B_{t+1}) \\ +\beta E_t \left[V(S_{t+1}, B_{t+1}; R_{t+1}^S, R_{t+1}^B) \right] \end{array} \right.
$$

First order conditions

$$
\begin{aligned}
B_{t+1} \quad &: \quad U'(C_t) = \beta E_t \left[V_B \left(S_{t+1}, B_{t+1}; R^S_{t+1}, R^B_{t+1} \right) \right] \\
S_{t+1} \quad &: \quad U'(C_t) = \beta E_t \left[V_S \left(S_{t+1}, B_{t+1}; R^S_{t+1}, R^B_{t+1} \right) \right]
\end{aligned}
$$

Differentiating the Bellman equation (1) with respect to S_t and B_t, using envelope conditions for V_B, and V_S, shifting one period forward and taking expectations, we get

$$
\begin{aligned}
E_t \left[V_B \left(S_{t+1}, B_{t+1}; R^S_{t+1}, R^B_{t+1} \right) \right] &= R^B_{t+1} \left[E_t U'(C_{t+1}) \right] \\
E_t \left[V_S (S_{t+1}, B_{t+1}; R^S_{t+1}, R^B_{t+1}) \right] &= E_t \left[R^S_{t+1} U'(C_{t+1}) \right]
\end{aligned}
$$

Notice: The return to the riskless asset R^B_{t+1} is known in period t.

First order conditions, cont.

- Plug back into FOCs:

$$U'(C_t) = R^B_{t+1}\beta E_t\left[U'(C_{t+1})\right] \quad (3)$$

$$U'(C_t) = \beta E_t\left[R^S_{t+1}U'(C_{t+1})\right] \quad (4)$$

- If$U'(C)$ = constant (risk neutral consumers), returns ought to be the same!
- Rearrange (3) and (4):

$$1 = R^B_{t+1}E_t\left\{\frac{\beta U'(C_{t+1})}{U'(C_t)}\right\} \quad (5)$$

$$1 = E_t\left\{R^S_{t+1}\frac{\beta U'(C_{t+1})}{U'(C_t)}\right\} \quad (6)$$

Stochastic discount factor

- $$m_{t,t+1} = \beta \frac{U'(C_{t+1})}{U'(C_t)} \tag{7}$$

  is the *stochastic discount* factor between periods t and $t+1$.
- *Stochastic discount factor* = the marginal rate of substitution of consumption from today to tomorrow = the marginal value of tomorrow's consumption in terms of today's consumption
- From today's perspective, tomorrow's consumption is uncertain, hence '*stochastic*'

Stochastic discount factor

- More generally
$$m_{t,t+j} = \beta^j \frac{U'(C_{t+j})}{U'(C_t)} \tag{8}$$
is the stochastic discount factor between periods t and $t+j$
- The stochastic discount factor is an essential element consumption based asset pricing. The stochastic discount factor is used in valuing revenue streams in different periods, and in different states of the world.

Equity premium

- Combining (5) and (6), and using the definition of the stochastic discount factor (7) we get

$$R^B_{t+1} E_t \left[m_{t,t+1} \right] = E_t \left[R^S_{t+1} m_{t,t+1} \right] \tag{9}$$

Next, since

$$\begin{aligned} Cov\left(R^S_{t+1}, m_{t,t+1}\right) &= E_t\left[R^S_{t+1} m_{t,t+1}\right] - E_t\left[m_{t,t+1}\right] E_t\left[R^S_{t+1}\right] \Leftrightarrow \\ E_t\left[R^S_{t+1} m_{t,t+1}\right] &= E_t\left[m_{t,t+1}\right] E_t\left[R^S_{t+1}\right] + Cov\left(R^S_{t+1}, m_{t,t+1}\right) \end{aligned}$$

the equation (9) can be re-written as

$$\begin{aligned} R^B_{t+1} E_t\left[m_{t,t+1}\right] &= E_t\left[m_{t,t+1}\right] E_t\left[R^S_{t+1}\right] + Cov\left(R^S_{t+1}, m_{t,t+1}\right) \Leftrightarrow \\ \underbrace{\frac{E_t\left[R^S_{t+1}\right] - R^B_{t+1}}{R^B_{t+1}}}_{\text{equity premium}} &= -\frac{Cov\left(R^S_{t+1}, m_{t,t+1}\right)}{R^B_{t+1} E_t\left[m_{t,t+1}\right]} \end{aligned} \tag{10}$$

Equity premium

- Next note that, by (5), i.e. Euler equation related to the choice of safe assets
$$1 = R^B_{t+1} E_t \left[m_{t,t+1} \right]$$
- Hence the equation (10) can be further simplified
$$\underbrace{\frac{E_t \left[R^S_{t+1} \right] - R^B_{t+1}}{R^B_{t+1}}}_{\text{equity premium}} = -Cov\left(R^S_{t+1}, m_{t,t+1} \right)$$
- Under risk neutrality $m_{t,t+1} = \beta =$ constant $\Rightarrow$ covariance term is zero and thus equity premium is also zero.
- When households are risk averse, they will hold risky assets only if they are offered a premium over the riskless asset.
  - well, actually the premium is positive only under certain assumptions; sometimes it can be even negative (see discussion below)
- This premium is proportional to the covariance of the asset's return with the *stochastic discount factor*

Equity premium

$$\underbrace{\frac{E_t\left\{R^S_{t+1}\right\} - R^B_{t+1}}{R^B_{t+1}}}_{\text{equity premium}} = -Cov\left(R^S_{t+1}, m_{t,t+1}\right)$$

- Positive equity premium $\Leftrightarrow$ negative covariance between returns from risky assets R^S_{t+1} and the stochastic discount factor $m_{t,t+1} = \frac{\beta U'(C_{t+1})}{U'(C_t)}$
- But the stochastic discount factor is negatively related to consumption growth (since $U''(C) < 0$)
  - high (low) consumption growth $\Rightarrow$ low (high) stochastic discount factor
- Hence: positive equity premium $\Leftarrow$consumption growth itself must be positively correlated with returns from risky asset.

Equity premium

- The higher the covariance between consumption and the returns of risky asset, the higher the risk premium must be in order for households to hold risky assets.
- Note: if the rate of return to some risky asset *covaries negatively* with consumption growth, the model predicts a *negative equity premium* for that particular asset, i.e. such an asset provides insurance against bad times.

Equity premium: some further intuition

- Assets that pay off when times are good and consumption levels are high, i.e. when the incremental value of additional consumption is low, are less desirable than those that pay off an equivalent amount when times are bad and additional consumption is both desirable and more highly valued.
- Hence, assets that pay off when times are good, must offer a higher expected return.
- Typically, stocks pay off (more) in good times (than in bad times) $\Rightarrow$ equity premium.

Next step: taking the model to the data

- The next task is to derive an equation that links the equity premium to statistics that can be measured from the data:
  i) the volatility of stock returns,
  ii) the volatility of consumption growth,
  iii) the correlation of stock returns and consumption growth.
- We use (5) and (6), and the fact that $U'(C) = C^{-\sigma}$ with CRRA utility function.
- We take (natural) logs on both sides and denote $\Delta C_{t+1} = \frac{C_{t+1}}{C_t}$.

$$0 = \ln \beta + \ln R^B_{t+1} + \ln E_t \left(\Delta C^{-\sigma}_{t+1} \right) \tag{11}$$

$$0 = \ln \beta + \ln E_t \left(R^S_{t+1} \Delta C^{-\sigma}_{t+1} \right) \tag{12}$$

Second-order Taylor approximation is needed

- Using first-order approximations would yield

$$\begin{aligned} E_t\left[r^S_{t+1}\right] &\approx \sigma E_t\left[c_{t+1}-c_t\right] \\ r^B_{t+1} &\approx \sigma E_t\left[c_{t+1}-c_t\right] \\ &\Rightarrow E_t\left[r^S_{t+1}\right] \approx r^B_{t+1} \end{aligned}$$

  where lower-case letters are log-deviations from the balanced growth path.
  But here the equity premium is nowhere to be seen!
- To study the equity premium, we need to take second-order Taylor approximations.
- Useful results: up to a second order approximation

$$\ln E_t\left[X_t\right] = E_t\left[\ln\left(X_t\right)\right] + \frac{1}{2} Var_t\left(\ln X_t\right)$$

  and

$$\ln E_t\left[X_t^\gamma\right] = \gamma E_t\left[\ln\left(X_t\right)\right] + \frac{1}{2}\gamma^2 Var_t\left(\ln X_t\right)$$

Second-order Taylor approximation, cont.

- By taking a second order Taylor approximation of $\ln E_t\left(\Delta C_{t+1}^{-\sigma}\right)$ around unconditional mean of ΔC_{t+1} in (11) we get the following expression:

$$
\begin{aligned}
0 &\approx \ln\beta + \ln R_{t+1}^{B} - \sigma E_t \ln(\Delta C_{t+1}) + \frac{1}{2}\sigma^2 Var_t(\ln \Delta C_{t+1}) \\
\ln R_{t+1}^{B} &\approx -\ln\beta + \sigma E_t \ln(\Delta C_{t+1}) - \frac{1}{2}\sigma^2 Var_t(\ln \Delta C_{t+1}) \qquad (13)
\end{aligned}
$$

- Notice: $\ln(\Delta C_{t+1})$ is percentage change (growth rate) of consumption, $E_t \ln(\Delta C_{t+1})$ is expected growth rate, and $Var_t(\ln \Delta C_{t+1})$ is the variance of growth rate.
- The unconditional mean $E\left[\Delta C_{t+1}\right] = \Delta\bar{C}$ is the long-run (or balanced growth path) growth rate of consumption.

Second-order Taylor approximation, cont.

- (12) can be further expressed as

$$
\begin{aligned}
0 &\approx \ln\beta + E_t\left(\ln R^S_{t+1} - \sigma\ln\Delta C_{t+1}\right) \\
&\quad + \frac{1}{2}Var(\ln R^S_{t+1} - \sigma\ln\Delta C_{t+1}) \\
E_t \ln R^S_{t+1} &\approx -\ln\beta + \sigma E_t\ln(\Delta C_{t+1}) \\
&\quad - \frac{1}{2}Var(\ln R^S_{t+1} - \sigma\ln\Delta C_{t+1}) \qquad (14) \\
&= -\ln\beta + \sigma E_t\ln(\Delta C_{t+1}) - \frac{1}{2}\sigma^2 Var_t(\ln\Delta C_{t+1}) \\
&\quad - \frac{1}{2}Var_t(\ln R^S_{t+1}) + \sigma Cov(\ln R^S_{t+1}, \ln\Delta C_{t+1})
\end{aligned}
$$

Second-order Taylor approximation, cont.

Finally, using the fact that

$$\ln E_t \left(R^S_{t+1} \right) \approx E_t \left\{ \ln R^S_{t+1} \right\} + \frac{1}{2} Var(\ln R^S_{t+1}) \tag{15}$$

and combining it with (14)

$$\begin{aligned} \ln E_t \left\{ R^S_{t+1} \right\} \quad &= \quad -\ln \beta + \sigma E_t \ln(\Delta C_{t+1}) - \frac{1}{2} \sigma^2 Var_t(\ln \Delta C_{t+1}) \\ &\quad + \sigma Cov(\ln R^S_{t+1}, \ln \Delta C_{t+1}) \end{aligned} \tag{16}$$

Equity premium, once again

- Now, combining (13) and (16) we get

$$\underbrace{\ln E_t \left\{ R^S_{t+1} \right\} - \ln R^B_{t+1}}_{\text{equity premium}} \approx \sigma Cov(\ln R^S_{t+1}, \ln \Delta C_{t+1})$$

- The equity premium depends on the covariance of stock returns $\ln R^S_{t+1}$ and consumption growth $\ln \Delta C_{t+1}$
  - If the performance of the stock market is positively correlated with consumption growth, the equity premium is positive
- The higher the coefficient of risk aversion σ, the more sensitive the equity premium is to the comovements of stock returns and consumption.

Equity premium puzzle

- In order for the model to explain equity premium, we need incredibly high curvature of relative risk aversion (σ). Micro evidence suggests that σ should be certainly less than 10. Perhaps in the range of 2-5. In many macro models $\sigma = 1$ (see e.g. Lecture 5-6).
- More formally, recall that

$$
\begin{aligned}
\underbrace{\ln E_t \left\{ R^S_{t+1} \right\} - \ln R^B_{t+1}}_{\text{equity premium}} &\approx \sigma Cov(\ln R^S_{t+1}, \ln \Delta C_{t+1}) \\
&= \sigma Corr(\ln R^S_{t+1}, \ln \Delta C_{t+1}) \sigma_{\ln R^S_{t+1}} \sigma_{\ln \Delta C_{t+1}}
\end{aligned}
$$

since $Corr\,(X, Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y}$

Equity premium puzzle, cont.

- Using U.S. data (sample 1889-1978, source: Mehra and Prescott (1985))

equity premium	$\sigma_{\ln R^S_{t+1}}$	$\sigma_{\ln \Delta C_{t+1}}$	$Corr(\ln R^S_{t+1}, \ln \Delta C_{t+1})$
6.2%	16.5%	3.6%	0.4

one would find that

$$0.062 \approx \sigma \times 0.4 \times 0.036 \times 0.165$$

$$\sigma = \frac{0.062}{0.4 \times 0.036 \times 0.165} = 26.1$$

Potential solution: habit persistence

- Constantinides (1990): Consumption growth rate appears to be too smooth to justify the average equity premium.
- Constantinides suggest that introducing habit persistence in consumption can resolve the problem.
- Habit persistence means that the past values of consumption matter for current utility.
- Current literature usually assumes the following utility function:

$$U(C_t, \widetilde{C}_{t-1}) = \frac{\left(C_t - b\widetilde{C}_{t-1}\right)^{1-\sigma}}{1-\sigma} \tag{17}$$

- $\widetilde{C}_{t-1}$ past average consumption in the economy (external habit persistence!), and $b \in (0, 1)$ captures the degree of habit persistence. That's "catching/keeping up with the Joneses".

Habit persistence, cont.

- Why does this help?
- Recall that

$$\underbrace{\frac{E_t\left\{R^S_{t+1}\right\}-R^B_{t+1}}{R^B_{t+1}}}_{\text{equity premium}} = -cov\left\{R^S_{t+1}, m_{t,t+1}\right\}$$

$$= -corr(R^S_{t+1}, m_{t,t+1}) \times \sigma_{R^S_{t+1}} \sigma_{m_{t,t+1}}$$

- where $m_{t,t+1} = \frac{\beta U'(C_{t+1})}{U'(C_t)}$ is the stochastic discount factor. The only way to make RHS large (at given properties of R^S_t) is to make sure that the stochastic discount factor $m_{t,t+1}$ is volatile without requiring consumption growth to be volatile.

Habit persistence, cont.

- Habit persistence does that. Combine eq. (17) with the definition of stochastic discount factor. Consider cases with $b = 1$ and $b = 0$:

$$\begin{aligned} b &= 1: \; m_{t,t+1} = \frac{\beta U'(C_{t+1})}{U'(C_t)} = \beta \left(\frac{C_{t+1} - C_t}{C_t - C_{t-1}} \right)^{-\sigma} \\ b &= 0: \; m_{t,t+1} = \frac{\beta U'(C_{t+1})}{U'(C_t)} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \end{aligned}$$

- When $b = 1$, relative changes in consumption growth matter. These can be large even if consumption growth itself is smooth. Thus the stochastic discount factor can be volatile at reasonable values of σ.

Firm acting in the interest of its shareholders

- Firm should act in the benefit of its shareholders, in our case households.
- Households receive dividends from the firms, and those dividends need to be somehow valued
- We find out that households "price" the dividends according to the stochastic discount factor.
- We will also derive an equation for Tobin's q and introduce investment adjustment costs (real rigidity)

Firm acting in the interest of its shareholders, cont.

- Let Q_t be the value of the ownership claim of the firm. What we want to get is an expression for Q_t which is consistent with the household's optimization problem.
- Recall the asset pricing equation:

$$1 = E_t \left[R_{t+1} m_{t,t+1} \right] \tag{18}$$

- Ownership claims worth Q_t in period t will deliver $Q_{t+1} + D_{t+1}$ in period $t+1$. Ex-post gross *return* is then $R_{t+1} = \frac{D_{t+1}+Q_{t+1}}{Q_t}$. Therefore, our asset pricing equation implies that:

$$1 = E_t \left\{ \left(\frac{D_{t+1} + Q_{t+1}}{Q_t} \right) m_{t,t+1} \right\}$$

- Re-arranging, we get

$$Q_t = \underbrace{E_t \left[m_{t,t+1} \left(D_{t+1} + Q_{t+1} \right) \right]}_{\substack{\text{expected discounted future dividends} \\ +\text{ expected discounted future value of the claim}}} \tag{19}$$

Firm acting in the interest of its shareholders, cont.

- (19) is recursive expression. Iterating forward over Q_t (bring (19) one period forward and plug it back into (19)):

$$Q_t = E_t \left[m_{t,t+1} \left(D_{t+1} + E_{t+1} \left[m_{t+1,t+2} \left(D_{t+2} + Q_{t+2} \right) \right] \right) \right] \quad (20)$$

  where

$$m_{t+1,t+2} = \beta \frac{U'(C_{t+2})}{U'(C_{t+1})}$$

  is the stochastic discount factor between periods $t+1$ and $t+2$

- Next notice that

$$\begin{aligned} m_{t,t+1} m_{t+1,t+2} &= \left(\beta \frac{U'(C_{t+1})}{U'(C_t)} \right) \left(\beta \frac{U'(C_{t+2})}{U'(C_{t+1})} \right) \\ &= \beta^2 \left(\frac{U'(C_{t+2})}{U'(C_t)} \right) = m_{t,t+2} \end{aligned}$$

Firm acting in the interest of its shareholders, cont.

- Then the equation (20) can be rewritten as

$$Q_t = E_t \left[m_{t,t+1} D_{t+1} + m_{t,t+2} \left(D_{t+2} + Q_{t+2} \right) \right] \tag{21}$$

- Here we have also taken on board the law of iterated expectations ($E_t \left[E_{t+1} \left[X_{t+2} \right] \right] = E_t \left[X_{t+2} \right]$)
- Iteration eventually delivers that the price of an ownership claim Q_t must be equal to expected discounted stream of dividends:

$$Q_t = E_t \left\{ \sum_{j=0}^{\infty} m_{t,t+j} D_{t+j} \right\}$$

Firm acting in the interest of its shareholders, cont.

- Important: *Discounting is based on household's stochastic discount factor* $m_{t,t+j}$.
- The discount factor is the marginal rate of substitution between consumption at date $t+i$ and consumption at date t.
- If households are risk neutral, $U'(.) = constant$. Then $m_{t,t+j} = \beta^j$ and

$$
\begin{aligned}
Q_t &= E_t\left\{\sum_{j=0}^{\infty}\beta^j D_{t+j}\right\} \\
Q_t &= E_t\left\{\sum_{j=0}^{\infty}\frac{D_{t+j}}{R^j}\right\}, \text{ since } R = \frac{1}{\beta}.
\end{aligned}
$$

where R is gross return of a riskless bond.

Firm's dynamic investment decision problem

- We now know that the price of the ownership claim depends on the expected discounted future stream of dividends.
- Let us now define dividends. In general,

$$\underbrace{D_t}_{\text{Dividends}} = \underbrace{\Pi(K_t, A_t)}_{\text{Profit function}} - \underbrace{(I_t + \Omega(I_t; K_t))}_{\substack{\text{Investment +}\\ \text{adjust. costs}}} \tag{22}$$

- $\Omega(I_t, K_t)$ captures costs of installing new capital (K). Because of this, the firm's profit maximization problem becomes dynamic. We assume that $\Omega_I(I_t, K_t) > 0$, $\Omega_K(I_t, K_t) < K_t$, $\Omega_{II}(I_t, K_t) > 0$, $\Omega_{KK}(I_t, K_t) > 0$.
- Increasing returns are ruled out by assuming that $\Pi(K_t, A_t)$ is quasi-concave.
- We abstract from decision over L_t: it is static one (there are no adjustment costs in labor).
  - $\Pi(K_t, A_t) = \Pi(K_t, A_t, W_t) = \max_{L_t} Y_t - W_t L_t$
- Price of *new* capital goods is normalized to one.

Detour: the case with no capital installation costs

- It is easy to conclude that with no installation cost we would have

$$Q_t = K_{t+1}$$

  - The value of the ownership claim Q_t is just the value of the period $t+1$ capital stock (remember that the value of K_{t+1} is set already in period t)
  - Also, the relative price of capital and the consumption good is 1 (since the consumption good can be transformed into capital one-to-one)

Detour: the case with no capital installation costs

- Dividends (in period $t+1$) are given by

$$D_{t+1} = \Pi(K_{t+1}, A_{t+1}) - I_{t+1}$$

- The gross real interest rate takes the familiar form

$$\begin{aligned} R_{t+1} &= \frac{D_{t+1} + Q_{t+1}}{Q_t} = \frac{\Pi(K_{t+1}, A_{t+1}) - I_{t+1} + K_{t+2}}{K_{t+1}} \\ &= \frac{\Pi(K_{t+1}, A_{t+1}) + (1-\delta)\, K_{t+1}}{K_{t+1}} \\ &= r^K_{t+1} + 1 - \delta \end{aligned}$$

where $r^K_{t+1} = \frac{\Pi(K_{t+1}, A_{t+1})}{K_{t+1}} = \frac{\alpha Y_{t+1}}{K_{t+1}} = \alpha A_{t+1} \left(\frac{K_{t+1}}{L_{t+1}} \right)^{\alpha - 1}$

recall that $\Pi(K_{t+1}, A_{t+1}) = \max_{L_{t+1}} Y_{t+1} - W_{t+1} L_{t+1}$

Firm's dynamic investment decision problem, cont.

- The firm's problem is

$$\begin{aligned}
& \max_{\{I_t, K_{t+1}\}} E_t \left\{ \sum_{j=1}^{\infty} m_{t,t+j} D_{t+j} \right\} \\
& s.t. \\
& D_t = \Pi(K_t, A_t) - [I_t + \Omega(I_t, K_t)] \\
K_{t+1} &= I_t + (1-\delta) K_t
\end{aligned} \tag{23}$$

where

$$m_{t,t+j} = \beta^j \frac{U'(C_{t+j})}{U'(C_t)}$$

is the stochastic discount factor between periods t and $t+j$.

Bellman equation

- Substitute (22) into objective function and substitute K_{t+1} away using constraint (23):

$$
\begin{aligned}
V(K_t, A_t, m_{t,t+1}) &= \max_{\{I_t\}} \{\Pi(K_t, A_t) - I_t - \Omega(I_t, K_t) \\
&\quad + E_t m_{t,t+1}[V(I_t + (1-\delta)K_t, A_{t+1}, m_{t+1,t+2})]\}
\end{aligned}
$$

  where

$$m_{t,t+1} = \beta \frac{U'(C_{t+1})}{U'(C_t)}$$

  is the stochastic discount factor from period t to period $t+1$.
- K_t is endogenous but predetermined state variable, A_t and $m_{t,t+1}$ are exogenous state variables, and I_t is control variable

First order condition

- $$I_t: \quad \underbrace{1+\Omega_I(I_t, K_t)}_{\substack{\text{replacement}\\ \text{costs of capital}}} = \underbrace{E_t[m_{t,t+1}V_K(K_{t+1}, A_{t+1}, m_{t+1,t+2})]}_{\substack{\text{discounted shadow}\\ \text{price of capital}\\ = \text{Tobin's q}}} \tag{24}$$
- .... so right hand side is the Tobin's q: the ratio of marginal value of capital *inside the firm* to the marginal value of capital *outside the firm*
- Due to adjustment costs, capital inside the firm (installed) is worth relatively more than capital outside the firm (not installed).
- There is a positive relationship between investment and Tobin's q.
- With *no capital installation costs* (i.e. if $\Omega_I(I_t, K_t) = 0$), Tobin's q is 1: the marginal value of capital is the same inside and outside the firm.

Envelope condition

- In order to find an expression for $V_K(K_{t+1}, A_{t+1})$, we utilize the envelope condition:

$$\begin{aligned}
V_K(K_t, A_t, m_{t,t+1}) &= \Pi_K(K_t, A_t) - \Omega_K(I_t, K_t) \\
&\quad + \underbrace{E_t m_{t,t+1} V_K(K_{t+1}, A_{t+1}, m_{t+1,t+2})}_{1+\Omega_I(I_t,K_t)}(1-\delta) \\
V_K(K_t, A_t, m_{t,t+1}) &= \Pi_K(K_t, A_t) - \Omega_K(I_t, K_t) + [1 + \Omega_I(I_t, K_t)](1-\delta)
\end{aligned}$$

- Lead this equation by one period and take expectations:

$$\begin{aligned}
E_t V_K(K_{t+1}, A_{t+1}, m_{t,t+1}) &= E_t \left[\Pi_K(K_{t+1}, A_{t+1}) - \Omega_K(I_{t+1}, K_{t+1})\right. \\
&\quad \left. + (1-\delta)[1 + \Omega_I(I_{t+1}, K_{t+1})]\right]
\end{aligned}$$

Plugging this back into FOC...

- By plugging this back into the first order condition (24) we obtain:

$$\begin{aligned} 1+\Omega_I(I_t,K_t) &= E_t[m_{t,t+1}\{\Pi_K(K_{t+1},A_{t+1})+1-\delta \\ &\quad -\Omega_K(I_{t+1},K_{t+1})+(1-\delta)\Omega_I(I_{t+1},K_{t+1})\}] \end{aligned} \tag{25}$$

- Notice that in the absence of the adjustment cost (25) boils down to

$$1=E_t\left[m_{t,t+1}\left\{\Pi_K(K_{t+1},A_{t+1})+(1-\delta)\right\}\right] \tag{26}$$

- Furthermore, if the production function has constant returns to scale $\Pi_K(K_{t+1},A_{t+1})=r^K_{t+1}$
- Next, recall that the consumer's Euler equation is given by $1=E_t\left[\beta R_{t+1}m_{t,t+1}\right]$
- Combining this with (26) gives

$$\begin{aligned} \Pi_K(K_{t+1},A_{t+1})+(1-\delta) &= r^K_{t+1}+(1-\delta)=R_{t+1} \\ &\Leftrightarrow r^K_{t+1}=r_{t+1}+\delta \end{aligned} \tag{27}$$

which is just the standard condition from static firm optimization.

Capital adjustment cost functions

- Quadratic:

$$\Omega(I_t, K_t) = \frac{b}{2}\left(\frac{I_t}{K_t} - \frac{\bar{I}}{\bar{K}}\right)^2 K_t \tag{28}$$

- Apply (28) in the first order condition (24)

$$\begin{aligned}
1 + \Omega_I(I_t, K_t) &= E_t[m_{t,t+1} V_K\left(K_{t+1}, A_{t+1}, m_{t+1,t+2}\right)] \\
1 + b\left(\frac{I_t}{K_t} - \frac{\bar{I}}{\bar{K}}\right) &= E_t[m_{t,t+1} V_K(K_{t+1}, A_{t+1}, m_{t+1,t+2})] \\
1 + b\left(\frac{I_t}{K_t} - \frac{\bar{I}}{\bar{K}}\right) &= E_t[m_{t,t+1} V_K(K_{t+1}, A_{t+1}, m_{t+1,t+2})] \\
\frac{I_t}{K_t} &= \frac{\bar{I}}{\bar{K}} - \frac{1}{b} + \frac{1}{b}\underbrace{E_t[m_{t,t+1} V_K(K_{t+1}, A_{t+1}, m_{t+1,t+2})]}_{\text{Tobin's q}}
\end{aligned}$$

- Investment is driven here solely by Tobin's q. Tobin's q is a sufficient statistic to predict investment.
  But there is a problem: We do not know Tobin's q!

Tobin's q

- Under certain assumptions (specifically, that adjustment cost function and profit function both exhibit constant returns to scale), it turns out that

$$\underbrace{V_K(K_t, A_t, m_{t,t+1})}_{\substack{\text{marginal value}\\ \text{of additional}\\ \text{unit of capital}}} = \underbrace{V(K_t, A_t, m_{t,t+1})/K_t}_{\substack{\text{average value of}\\ \text{unit of capital}}} \tag{29}$$

- ... and

$$q_t = \frac{E_t\left[m_{t+1,t} V(K_{t+1}, A_{t+1}, m_{t+1,t+2})\right]}{K_{t+1}}$$

Tobin's q, cont.

- Average value of the firm can be computed using stock market data.
- One often used measure of Tobin's q is

$$q = \frac{\text{market value of equity} + \text{market value of liabilities}}{\text{book value of equity} + \text{book value of liabilities}}$$

- One could then estimate:

$$\frac{I_t}{K_t} = \beta_0 + \beta_1 q_t + \zeta \underbrace{X_t}_{\text{other variables}}$$

- In practice $\zeta \neq 0$. Other financial variables explain investment. Also, in practice investment depends on its own past values.

Alternative specification of adjustment costs

- Another way of introducing investment adjustment costs is to rewrite capital accumulation equation as

$$K_{t+1} = [1 - S(.)]I_t + (1-\delta)K_t$$

where $S(.)$ denotes the investment adjustment cost function.

- Christiano, Eichenbaum and Evans (2004) propose $S(.) = S(\frac{I_t}{I_{t-1}})$, so that the cost involves changing the level of investment. One "convenient" formulation is

$$S\left(\frac{I_t}{I_{t-1}}\right) = \frac{b}{2}\left[\exp\left\{\frac{I_t}{I_{t-1}} - 1\right\} + \exp\left\{-\left(\frac{I_t}{I_{t-1}} - 1\right)\right\} - 2\right] \tag{30}$$

- Here $S(1) = S'(1) = 0$, and $S''(1) = b$. Note: it is also easy to embed this in a model with balanced growth path.

Last look at the envelope condition

- Recall from previous analysis that

$$
\begin{aligned}
V_K(K_t, A_t, m_{t,t+1}) &= \Pi_K(K_t, A_t) - \Omega_K(I_t, K_t) \qquad (31) \\
&\quad + E_t\left[m_{t,t+1}(1-\delta) V_K(K_{t+1}, A_{t+1}, m_{t+1,t+2})\right]
\end{aligned}
$$

- This can be iterated forward to deliver

$$
\begin{aligned}
& V_K(K_t, A_t, m_{t,t+1}) \\
= \quad & E_t\left(\sum_{i=0}^{\infty} m_{t,t+i}\,(1-\delta)^i\left[\Pi_K(K_{t+i}, A_{t+i}) - \Omega_K(I_{t+i}, K_{t+i})\right]\right)
\end{aligned}
$$

- That is, the marginal value of capital is equal to discounted stream of marginal profits minus the capital adjustment costs.

Embedding the firm's dynamic problem into a RBC model

Using the expression of the stochastic discount factor, $m_{t,t+1} = \beta \frac{U'(C_{t+1})}{U'(C_t)}$, the first-order condition of firm optimization (25) can be rewritten as

$$1 = E_t \left[\beta R_{t+1} \frac{U'(C_{t+1})}{U'(C_t)} \right] \tag{32}$$

where

$$R_{t+1} = \frac{r_{t+1}^K - \Omega_K(I_{t+1}, K_{t+1}) + (1-\delta)\, q_{t+1}}{q_t} \tag{33}$$

is the real interest rate, and

$$r_{t+1}^K = \Pi_K(K_{t+1}, A_{t+1}) = \alpha Y_t / K_t = \alpha A_{t+1} \left(\frac{K_{t+1}}{L_{t+1}} \right)^{\alpha-1} \tag{34}$$

is the marginal product of capital, and

$$q_t = 1 + \Omega_I(I_t, K_t) \tag{35}$$

is the Tobin's q. Notice that the equation (32) is just the standard Euler equation.

Embedding the firm's dynamic problem into a RBC model

- The remaining equations characterizing the equilibrium include
- ... the labor market equilibrium condition

$$\frac{v'(L_t)}{U'(C_t)} = \underbrace{(1-\alpha) A_t \left(\frac{K_{t+1}}{L_{t+1}}\right)^{\alpha}}_{W_t} \tag{36}$$

(where $v'(L_t)$ is the marginal disutility from working)

- ... the aggregate resource constraint

$$C_t + I_t + \Omega(I_t, K_t) = A_t K_t^{\alpha} L_t^{1-\alpha} \tag{37}$$

- ... the law of motion of the capital stock

$$K_{t+1} = (1-\delta) K_t + I_t \tag{38}$$

- ... and the law of motion of TFP

$$\ln(A_t) = (1-\rho)\ln(\bar{A}) + \rho A_{t-1} + \epsilon_t \tag{39}$$

Embedding the firm's dynamic problem into a RBC model

- The dynamic equilibrium of the model is characterized by the equations (32)-(39): 8 equations for 8 variables $(C_t, I_t, K_t, L_t, R_t, r_t^K, q_t, A_t)$.
- Evidently, we could plug the expressions (33), (34) and (35) into (32): this would result in 5 equations for 5 variables $(C_t, I_t, K_t, L_t, A_t)$.
- Note that the dynamic equilibrium of the model could be also derived by solving the planner's problem:

$$\max E_0 \sum_{t=0}^{\infty} U(C_t) - v(L_t)$$

subject to (37), (38) and (39)

The volatility of investment

- Standard RBC models often produce too volatile investment (see for example the results from Hansen's model in Lecture 5-6).
- Convex capital adjustment cost makes investment less volatile and helps to square the model with the data.
- Another assumption that may help in this respect is time-varying capital utilization rate (u_t)

$$Y_t = A_t \left(u_t K_t\right)^{\alpha} L_t^{1-\alpha}$$

  - In good times (positive TFP shock) capital is used more intensively $\Rightarrow$ there is less need for new investment
- In addition, variable capital utilization amplifies the effects of TFP shocks
- ... and raises questions about the appropriateness of the Solow residual as a measure of TFP (see Lecture 5-6)